

# A REMARK ON THE OMORI-YAU MAXIMUM PRINCIPLE

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ABSTRACT. A Riemannian manifold  $M$  is said to satisfy the Omori-Yau maximum principle if for any  $C^2$  bounded function  $g : M \rightarrow \mathbb{R}$  there is a sequence  $x_n \in M$ , such that  $\lim_{n \rightarrow \infty} g(x_n) = \sup_M g$ ,  $\lim_{n \rightarrow \infty} |\nabla g(x_n)| = 0$  and  $\limsup_{n \rightarrow \infty} \Delta g(x_n) \leq 0$ . It is shown that if the Ricci curvature does not approach  $-\infty$  too fast the manifold satisfies the Omori-Yau maximum principle. This improves earlier necessary conditions. The given condition is quite optimal.

## 0. INTRODUCTION

**Definition.** A Riemannian manifold  $M$  is said to satisfy the Omori-Yau maximum principle if for any  $C^2$  function  $g : M \rightarrow \mathbb{R}$  which is bounded from above and for any  $\epsilon > 0$  there is a point  $x_\epsilon \in M$ , such that  $|g(x_\epsilon) - \sup_M g| < \epsilon$ ,  $|\nabla g(x_\epsilon)| < \epsilon$  and  $\Delta g(x_\epsilon) < \epsilon$ .

This principle has turned out to be very useful in differential geometry and received considerable attention recently. A necessary condition in terms of the Ricci curvature for a manifold to satisfy this principle was first proved by Omori in [O] and later generalized by Yau [Y]. It states that if the Ricci curvature is bounded from below then the manifold satisfies the Omori-Yau maximum principle.

This was improved upon by Ratto, Rigoli and Setti in [RRS, Theorem 2.3].

**Theorem (Ratto-Rigoli-Setti).** Let  $M^n$  be a complete Riemannian manifold,  $p \in M^n$  be a fixed point and  $r(x)$  be the distance function from  $p$ . Let us assume that away from the cut locus of  $p$  we have

$$\text{Ric}(\nabla r, \nabla r) \geq -(n-1)BG^2(r),$$

where  $B > 0$  is some constant and  $G(t)$  has the following properties:

- (i)  $G(0) = 1, \quad G' \geq 0$
- (ii)  $\int_0^\infty \frac{dt}{G(t)} = \infty$
- (iii)  $\frac{d^{2k+1}}{dt^{2k+1}} \sqrt{G}(0) = 0 \quad \text{for all } k \in \mathbb{N}$
- (iv)  $\limsup_{t \rightarrow \infty} \frac{t\sqrt{G(\sqrt{t})}}{\sqrt{G(t)}} < \infty.$

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*Then  $M^n$  satisfies the Omori-Yau maximum principle.*

The goal of the present note is to improve the necessary condition given in [RRS]. The actual statement is given as a Corollary. Basically we remove the last two conditions on the function  $G(t)$ , which turned out not to be essential.

Another interesting necessary condition, requiring the existence of an exhaustion function with certain properties, was given by Kim and Lee in [KL]. Interestingly there is an alternative proof by Kim and Lee of the Ratto-Rigoli-Setti result in [KL] which is still using these extra conditions.

The proof uses the same method we used in an earlier paper [B].

**Theorem.** *Let  $M^n$  be a complete Riemannian manifold,  $p \in M^n$  be a fixed point and  $r(x)$  be the distance function from  $p$ . Let us assume that*

$$\Delta r(x) \leq G(r(x))$$

*for all  $x \in M^n$  where  $r$  is smooth and  $r(x) > 1$ , where  $G(t)$  has the following properties:*

$$G \geq 1, \quad G' \geq 0, \quad \text{and} \quad \int_0^\infty \frac{dt}{G(t)} = \infty.$$

*Then  $M^n$  satisfies the Omori-Yau maximum principle.*

As a consequence we have the following.

**Corollary.** *Let  $M^n$  be a complete Riemannian manifold,  $p \in M^n$  be a fixed point and  $r(x)$  be the distance function from  $p$ . Let us assume that away from the cut locus of  $p$  we have*

$$\text{Ric}(\nabla r, \nabla r) \geq -G^2(r),$$

*where  $G(t)$  has the following properties:*

$$G \geq 1, \quad G' \geq 0, \quad \text{and} \quad \int_0^\infty \frac{dt}{G(t)} = \infty.$$

*Then  $M^n$  satisfies the Omori-Yau maximum principle.*

The main condition on the function  $G(t)$  in the Corollary and in the Ratto-Rigoli-Setti Theorem is the same ( $\int 1/G(t) = \infty$ ) but there are additional technical conditions imposed on the function  $G(t)$  in the later Theorem. In this respect Corollary can be considered as a refinement of the Ratto-Rigoli-Setti Theorem.

Let us mention that this condition is quite optimal. If  $\int_0^\infty 1/G(t)dt < \infty$ , there are manifolds with  $\Delta r \leq G(r)$  for which the Omori-Yau maximum principle does not apply. The details can be found in Section 3.

## 1. PROOF OF THE THEOREM

*Proof of the Theorem.* Set  $L = \sup g$  and let us assume that  $g < L$  at every point of  $M$ . Otherwise  $g$  assumes its maximum at some point and that point trivially satisfies the conditions of the Definition for all  $\epsilon > 0$ .

Define the function  $F(t)$  as

$$F(t) = e^{\int_0^t \frac{1}{G(s)} ds}.$$

Then clearly:  $F \geq 1$ ,  $F$  is strictly increasing and  $\lim_{t \rightarrow \infty} F(t) = \infty$ .

For any  $\epsilon < \min\{1, L - \sup\{g(x) : r(x) < 1\}\}$  define the function  $h_\lambda : M \rightarrow \mathbb{R}$  as

$$h_\lambda(x) = \lambda F(r(x)) + L - \epsilon.$$

Since  $F(r(x)) \geq 1$ , for  $\lambda > \epsilon$  we have

$$h_\lambda(x) > L > g(x) \quad \text{for all } x \in M.$$

Define  $\lambda_0$  as

$$\lambda_0 = \inf\{\lambda : h_\lambda(x) > g(x) \text{ for all } x \in M\}.$$

Since  $\sup g = L$  it is easy to see that  $\lambda_0 > 0$  and  $h_{\lambda_0}(x) \geq g(x)$  for all  $x \in M$ .

We claim that there is a point  $x_\epsilon \in M$  such that  $h_{\lambda_0}(x_\epsilon) = g(x_\epsilon)$ .

This will follow from the observation that if  $h_\lambda(x) > g(x)$  for all  $x \in M$ , then there is a  $\lambda' < \lambda$  such that  $h_{\lambda'}(x) > g(x)$  for all  $x \in M$ . To show this we argue as follows.

Let  $r_0$  be large enough such that  $h_\lambda(x) > L+1$  for  $r(x) > r_0$ . Since  $\lim_{r \rightarrow \infty} F(r) = \infty$  such  $r_0$  must exist. The set  $\{x \in M : r(x) \leq r_0\}$  is compact, therefore  $h_\lambda(x) > g(x)$  for all  $x \in M$  implies that there is a  $\lambda' < \lambda$  such that  $h_{\lambda'}(x) > g(x)$  for all  $x \in M : r(x) \leq r_0$ . Choosing  $\lambda'$  sufficiently close to  $\lambda$  we can achieve that  $h_{\lambda'}(x) > L$  for  $r(x) = r_0$ . Since  $F$  is increasing we obtain that  $h_{\lambda'}(x) > L$  for  $r(x) \geq r_0$ . Combining this with the previous remark we have  $h_{\lambda'}(x) > g(x)$  for all  $x \in M$ .

Next, we have to show that  $h_{\lambda_0}$  is smooth at  $x_\epsilon$ . The argument is exactly the same as the argument in [B], but we include it at the end of this proof for the convenience of the reader.

Once we established the smoothness of  $h_{\lambda_0}$  at  $x_\epsilon$ , the rest of the argument is straight forward.

From the definition of  $F$  and from the fact that  $G' \geq 0$  we have

$$F' = \frac{F}{G} \quad \text{and} \quad F'' = \frac{F'}{G} - \frac{FG'}{G^2} \leq \frac{F}{G^2}.$$

From the fact that  $g(x_\epsilon) = \lambda_0 F(r(x_\epsilon)) + L - \epsilon < L$  we conclude that

$$L - g(x_\epsilon) \leq \epsilon, \tag{1.1}$$

moreover

$$\lambda_0 F(r(x_\epsilon)) < \epsilon \quad \text{hence} \quad \lambda_0 < \frac{\epsilon}{F(r(x_\epsilon))} < \epsilon. \tag{1.2}$$

Since

$$h_{\lambda_0}(x) \geq g(x), \quad \text{and} \quad h_{\lambda_0}(x_\epsilon) = g(x_\epsilon),$$

we have

$$\nabla g(x_\epsilon) = \nabla h_{\lambda_0}(x_\epsilon) \quad \text{and} \quad \Delta h_{\lambda_0}(x_\epsilon) \geq \Delta g(x_\epsilon).$$

Taking into consideration (1.2), the definition of  $F$ , the fact that  $|\nabla r| = 1$  and the assumption that  $G(r) \geq 1$ , the first equality above yields

$$|\nabla g(x_\epsilon)| = |\lambda_0 F'(r(x_\epsilon)) \nabla r(x_\epsilon)| = \frac{\epsilon}{F(r)} \cdot \frac{F'(r)}{G(r)} < \epsilon. \tag{1.3}$$

For the Laplace of  $h_{\lambda_0}$  we have

$$\begin{aligned} \Delta g(x_\epsilon) &\leq \Delta h_{\lambda_0}(x_\epsilon) = \lambda_0 \left( F'(r(x_\epsilon)) \Delta r(x_\epsilon) + F''(r(x_\epsilon)) |\nabla r(x_\epsilon)|^2 \right) \leq \\ &\leq \frac{\epsilon}{F} \left( \frac{F}{G} \Delta r + \frac{F}{G^2} \right) \leq 2\epsilon. \end{aligned} \quad (1.4)$$

The inequalities (1.1), (1.3) and (1.4) show that the point  $x_\epsilon$  satisfies the conditions in the Definition.

Finally, we have to show that  $h_{\lambda_0}$  is smooth at  $x_\epsilon$ . Since  $h_\lambda(x) = \lambda F(r(x)) + L - \epsilon$  it is enough to show that  $r$  is smooth at  $x_\epsilon$ . If not, then  $x_\epsilon$  must be on the cut locus of  $p$ . In this case we have two possibilities. Either there are two distinct minimizing geodesic segments  $\gamma_1, \gamma_2 : [0, t_0] \rightarrow M$  joining  $p$  to  $x_\epsilon$ , or there is a geodesic segment  $\gamma : [0, t_0] \rightarrow M$  from  $p$  to  $x_\epsilon$  along which  $x_\epsilon$  is conjugate to  $p$ .

In both cases we have

$$t_0 = r(x_\epsilon).$$

Let us start with the first case. Let  $w = \gamma_1'(t_0)$  and  $v = \gamma_2'(t_0)$ . Since  $\gamma_1$  and  $\gamma_2$  are distinct segments we have  $w \neq v$ . The functions  $t \rightarrow r(\gamma_i(t))$  are differentiable on  $(0, t_0)$  (for  $i = 1, 2$ ) and they have a left-derivative at  $t_0$ .

From the fact that  $h_{\lambda_0} \geq g$  and  $h_{\lambda_0}(x_\epsilon) = g(x_\epsilon)$  we have

$$\liminf_{s \rightarrow 0^+} \frac{h_{\lambda_0}(\gamma_2(t_0 + s)) - h_{\lambda_0}(\gamma_2(t_0))}{s} \geq D_v g(x_\epsilon),$$

where  $D_v g(x_\epsilon)$  denotes the directional derivative of  $g$  at the point  $x_\epsilon$  in the direction of  $v$ . Moreover since  $g$  is smooth and  $h_{\lambda_0}$  has a directional derivative at  $x_\epsilon$  in the direction of  $-v$ , we also have

$$-\lambda_0 F'(r(x_\epsilon)) = D_{-v} h_{\lambda_0}(x_\epsilon) \geq D_{-v} g(x_\epsilon) = -D_v g(x_\epsilon).$$

This yields

$$D_v g(x_\epsilon) \geq \lambda_0 F'(r(x_\epsilon)). \quad (1.5)$$

Combining this with the above inequality we obtain

$$\liminf_{s \rightarrow 0^+} \frac{h_{\lambda_0}(\gamma_2(t_0 + s)) - h_{\lambda_0}(\gamma_2(t_0))}{s} \geq \lambda_0 F'(r(x_\epsilon)).$$

Taking into account the special form of  $h_{\lambda_0}$  we have

$$\liminf_{s \rightarrow 0^+} \frac{r(\gamma_2(t_0 + s)) - r(\gamma_2(t_0))}{s} \geq 1. \quad (1.6)$$

This will lead to a contradiction. Since  $v \neq w$ , there is a  $0 < c < 1$  depending only on the angle of  $v$  and  $w$ , such that

$$r(\gamma_2(t_0 + s)) < t_0 + cs, \quad (1.7)$$

for a small enough  $s > 0$ .

One can see this by connecting the point  $\gamma_1(t_0 - s)$  to  $\gamma_2(t_0 + s)$  by a geodesic segment. Since  $\gamma_1$  and  $\gamma_2$  are different there is a  $0 < c_1 < 1$  such that for a small enough  $s > 0$  we have  $\text{dist}(\gamma_1(t_0 - s), \gamma_2(t_0 + s)) < c_1 2s$  and this implies (1.7). Since  $r(x_\epsilon) = r(\gamma_2(t_0)) = t_0$  it is easy to see that (1.6) and (1.7) are in direct contradiction.

We now turn our attention to the second case. Since  $\gamma$  is distance minimizing between  $p$  and  $x_\epsilon$  the distance function  $r$  is smooth at  $\gamma(t)$  for  $0 < t < t_0$ . Set  $m(t) = \Delta r(\gamma(t))$ . Then  $m(t)$  is also smooth on the interval  $(0, t_0)$  and since  $\gamma(t_0)$  is conjugate to  $p = \gamma(0)$  along  $\gamma$  we have

$$\lim_{t \rightarrow t_0^-} m(t) = -\infty. \quad (1.8)$$

Since  $\lambda_0 > 0$ , from (1.5) we conclude that  $D_v g(x_\epsilon) > 0$ , that is  $\nabla g(x_\epsilon) \neq 0$ . This implies that the level surface  $H = \{x \in M : g(x) = g(x_\epsilon)\}$  is a smooth hypersurface near  $x_\epsilon$ . Denote by  $H_s$  the surface parallel to  $H$  and passing through the point  $\gamma(t_0 - s)$  for some  $s > 0$ . Again, since  $H$  is smooth near  $x_\epsilon$  the surface  $H_s$  will also be smooth near  $\gamma(t_0 - s)$  for a small enough  $s > 0$ .

It is now clear from (1.8) that for some small  $s > 0$  we have

$$m(t_0 - s) < \text{trace of the 2nd fundamental form of } H_s \text{ at } \gamma(t_0 - s),$$

where the second fundamental form of  $H_s$  at  $\gamma(t_0 - s)$  is taken in the direction of  $\gamma'(t_0 - s)$ .

Taking into account that  $m(t_0 - s)$  is the trace of the 2nd fundamental form of the geodesic ball  $B_p(t_0 - s)$  around  $p$  at the point  $\gamma(t_0 - s)$  (with respect to the same normal vector  $\gamma'(t_0 - s)$ ) we conclude that there has to be a point  $q_s \in H_s$ , sufficiently close to  $\gamma(t_0 - s)$ , that lies inside  $B_p(t_0 - s)$ . This means that

$$r(q_s) < t_0 - s.$$

Since  $H_s$  is parallel to  $H$  we have a point on  $q \in F$  such that  $\text{dist}(q_s, q) = s$ . Combining this with the above inequality we have

$$r(q) < t_0 = r(x_\epsilon).$$

Taking into account that  $F$  is strictly increasing we obtain

$$h_{\lambda_0}(q) = \lambda_0 F(r(q)) + L - \epsilon < \lambda_0 F(r(x_\epsilon)) + L - \epsilon = h_{\lambda_0}(x_\epsilon) = g(x_\epsilon) = g(q).$$

This leads to a contradiction since  $h_{\lambda_0} \geq g$  on  $M$ .

## 2. PROOF OF THE COROLLARY

Let  $q \in M$  be a point away from the cut locus of  $p$  and  $\gamma$  be a geodesic segment parameterized by arc length connecting  $p$  to  $q$ . Set  $m(t) = \Delta r(\gamma(t))$  and  $R(t) = \text{Ricc}(\gamma'(t), \gamma'(t))$ . Then it is well known that  $m(t)$  satisfies the Riccati inequality along  $\gamma$ . Taking into consideration the condition on the Ricci curvature we have

$$m'(t) \leq -R(t) - \frac{m^2(t)}{n-1} \leq G^2(t) - \frac{m^2(t)}{n-1}.$$

This implies that  $m$  is decreasing as long as  $m > \sqrt{n-1}G$  and a simple argument shows that

$$m(t) < (\sqrt{n-1} + 1)G,$$

for all  $t > t_0$ , where  $t_0$  is a sufficiently large constant, independent of  $G$ .

This yields

$$\Delta r < (\sqrt{n-1} + 1)G \quad \text{if } r > t_0,$$

for points that are not on the cut locus of  $p$ . Since  $(\sqrt{n-1} + 1)G$  satisfies the conditions in the Theorem the proof of the corollary is complete.

### 3. AN EXAMPLE

In this section we sketch an example, that shows that the condition in the Theorem is quite optimal. Let  $M^n$  be a Hadamard manifold that is rotationally symmetric around  $p \in M^n$ .

Let  $r$  be the distance function from  $p$  and assume that  $\Delta r(x) > G(r)$  for all  $x \in M^n$ , where  $G$  satisfies the conditions:

$$G \geq 1, \quad G' \geq 0, \quad \text{and} \quad \int_0^\infty \frac{dt}{G(t)} < \infty.$$

Then there is a bounded function  $h : M \rightarrow \mathbb{R}$  which shows that the manifold  $M^n$  does not satisfy the Omori-Yau maximum principle. To construct  $h$  we need the following lemma.

**Lemma.** *Let  $G : [0, \infty) \rightarrow \mathbb{R}$  be a function satisfying the conditions:*

$$G \geq 1, \quad G' \geq 0, \quad \text{and} \quad \int_0^\infty \frac{dt}{G(t)} < \infty.$$

*Then there is a function  $H : [0, \infty) \rightarrow \mathbb{R}$  such that*

$$H \geq 1/2, \quad H' \geq 0, \quad 2H \leq G, \quad H' \leq H^2 \quad \text{and} \quad \int_0^\infty \frac{dt}{H(t)} < \infty.$$

First we construct the function  $h : M^n \rightarrow \mathbb{R}$  and give the proof of the Lemma later.

Let

$$h(x) = \int_0^{r(x)} \frac{dt}{H(t)}.$$

The last condition on  $H$  in the Lemma implies that  $h$  is bounded from above. A simple computation shows that

$$\Delta h = \frac{\Delta r}{H} - \frac{H'}{H^2} |\nabla r|^2.$$

Since  $\Delta r > G(r) \geq 2H(r)$ ,  $|\nabla r| = 1$  and  $H' \leq H$  we have

$$\Delta h > 2 - 1 = 1.$$

This clearly shows that the manifold  $M^n$  does not satisfy the Omori-Yau maximum principle.

All that remains is to prove the Lemma.

*Proof of Lemma.* Let  $A \subset (0, \infty)$  be defined as

$$A = \{t > 0 : \frac{G'(t)}{2} > \left(\frac{G(t)}{2}\right)^2\}.$$

It is an open set therefore

$$A = \cup I_n,$$

where  $I_n = (t_n, s_n)$  are disjoint open intervals.

This is the set where  $G/2$  grows too fast. We obtain  $H$  by modifying  $G/2$  on a slightly larger set so that it will never grow too fast, that is  $H' \leq H^2$ .

For a given  $n$  define the function  $k_n(t)$  to be

$$k_n(t) = \frac{1}{a_n - t},$$

where  $a_n$  is chosen such that  $k_n(t_n) = G(t_n)/2$ . Then we have

$$k_n(t_n) = \frac{G(t_n)}{2}, \quad k'_n(t) = k_n^2(t) \quad \text{and} \quad \frac{G'(t)}{2} > \left(\frac{G(t)}{2}\right)^2 \quad \text{for} \quad t \in (t_n, \min\{s_n, a_n\}).$$

This implies that

$$k_n(t) < \frac{G(t)}{2} \quad \text{for} \quad t \in (t_n, \min\{s_n, a_n\}).$$

Let  $v_n > t_n$  be the first point where  $k_n(v_n) = G(v_n)/2$ . Such point must exist since  $\lim_{t \rightarrow a_n} k_n(t) = \infty$ . Therefore we have  $t_n < s_n < v_n < a_n$  and as a result  $J_n = (t_n, v_n) \supset I_n$ .

The intervals  $I_n$  are all disjoint but  $J_n$  are not necessarily disjoint intervals. However if  $J_n \cap J_m \neq \emptyset$ , then either  $J_n \subset J_m$  or  $J_m \subset J_n$ . This follows simply from the way the intervals  $J_n$  were constructed and from the fact that the graphs of the functions  $1/(a-t)$ ,  $t < a$  and  $1/(b-t)$ ,  $t < b$  are translates of each other.

Therefore we can select a pairwise disjoint family of intervals  $J_{n_i}$  such that  $B = \cup J_n = \cup J_{n_i}$ . To simplify the notation without loss of generality we can assume that the intervals  $J_n$  are already pairwise disjoint.

We can now define the function  $H(t)$  as follows

$$H(t) = \begin{cases} \frac{G(t)}{2} & \text{if } t \notin B = \cup J_n \\ \frac{1}{a_n - t} & \text{if } t \in J_n. \end{cases}$$

It is clear from the construction that  $H$  satisfies the first four properties in the Lemma. It remains to show that it will satisfy the remaining property

$$\int_0^\infty \frac{dt}{H(t)} < \infty. \tag{3.1}$$

We can write

$$\int_0^\infty \frac{dt}{H(t)} = \int_B \frac{dt}{H(t)} + \int_{\mathbb{R}^+ - B} \frac{dt}{H(t)}.$$

The second integral is clearly finite since

$$\int_{\mathbb{R}^+ - B} \frac{dt}{H(t)} = \int_{\mathbb{R}^+ - B} \frac{2dt}{G(t)} < \infty.$$

The first integral can be computed as follows

$$\int_B \frac{dt}{H(t)} = \int_{\cup J_n} \frac{dt}{H(t)} = \sum_{n=1}^\infty \int_{t_n}^{v_n} a_n - t \, dt = \frac{1}{2} \sum_{n=1}^\infty (a_n - t_n)^2 - (a_n - v_n)^2.$$

From the construction of the intervals  $J_n$  and the function  $H$  one obtains that

$$a_n - t_n = \frac{2}{G(t_n)} \quad \text{and} \quad a_n - v_n = \frac{2}{G(v_n)}.$$

To show that the infinite sum above is finite it is enough to show that any partial-sum is bounded by a fixed constant. For this reason consider the sum

$$\sum_{n=1}^m (a_n - t_n)^2 - (a_n - v_n)^2 = \sum_{n=1}^m \left( \frac{2}{G(t_n)} \right)^2 - \left( \frac{2}{G(v_n)} \right)^2.$$

By rearranging the terms if necessary, without loss of generality we can assume that

$$t_1 < v_1 < t_2 < v_2 < \dots < t_n < v_n < t_{n+1} < v_{n+1} < \dots < t_m < v_m.$$

Taking into consideration that  $G(t)$  is an increasing function, we obtain that

$$\sum_{n=1}^m (a_n - t_n)^2 - (a_n - v_n)^2 = \sum_{n=1}^m \left( \frac{2}{G(t_n)} \right)^2 - \left( \frac{2}{G(v_n)} \right)^2 < \left( \frac{2}{G(t_1)} \right)^2.$$

This shows that the above sum is finite, which in turn proves (3.1). This completes the proof of the lemma.

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